

Technical Comments

TECHNICAL COMMENTS are brief discussions of papers previously published in this journal. They should not exceed 1500 words (where a figure or table counts as 200 words). The author of the previous paper is invited to submit a reply for publication in the same issue as the Technical Comment. These discussions are published as quickly as possible after receipt of the manuscripts. Neither AIAA nor its Editors are responsible for the opinions expressed by the authors.

Comment on “Precision Free-Inertial Navigation with Gravity Compensation by an Onboard Gradiometer”

Thomas H. Kerr III*

TeK Associates, Lexington, Massachusetts 02420-5336

DOI: 10.2514/1.29808

AN APPARENT weakness in the arguments within the derivation of [1] (Appendix) is identified using explicit numerical examples which further demonstrate that the results are of limited benefit.

Our prior experience in specifying *linear system realizations* [2,3] alerted us to an apparent problem with the new alternative procedure offered in [1] (Appendix) as a potentially more straightforward way to achieve a multi-input multi-output linear system realization from a matrix power spectrum or, equivalently, from a *given* matrix correlation function by explicitly delineating both the structure and parameter values of an underlying white noise driven linear time-invariant state variable model that provides such a vector random process as its output.

The arguments for the derivation of the matrix Lyapunov equation (which the variance/cross-covariance matrix satisfies[†]) ([5], pp. 222–226) are quite familiar to many analysts but fall short in Jekeli's attempt to extend them in its overly concise but appealing result, where Eq. A4 equates a function of one variable on the left-hand side[‡] to a function of two variables on the right-hand side as an obvious impossibility.

Among the beneficial results offered in [2,3], is a nondegenerate *statistically stationary* two-channel numerical example[§]:

$$R_{yy}(\tau) = \begin{bmatrix} \frac{1}{6}e^{-2|\tau|} + \frac{1}{6}e^{-|\tau|} & \vdots & \frac{1}{4}e^{-2|\tau|} \\ \dots & \dots & \dots \\ \frac{1}{4}e^{-2|\tau|} & \vdots & \frac{1}{2}e^{-|\tau|} \end{bmatrix}$$

$$\Leftrightarrow S_{yy}(s) = \begin{bmatrix} \frac{2-s^2}{(4-s^2)(1-s^2)} & \frac{1}{(4-s^2)} \\ \frac{1}{(4-s^2)} & \frac{1}{(1-s^2)} \end{bmatrix} = W^T(-s)W(s) \quad (1)$$

where these second-order statistics correspond to a demonstrable closed-form solution both for the intermediate (nonunique [3]) matrix spectral factorization (MSF):

$$W^T(s) \equiv \begin{bmatrix} \frac{-s-(\sqrt{7}/2)}{(2+s)(1+s)} & \frac{-1/2}{(2+s)(1+s)} \\ \frac{-s-(\sqrt{7}/2)}{(2+s)(1+s)} & \frac{3/2}{(2+s)(1+s)} \end{bmatrix} = H(sI - F)^{-1}G \quad (2)$$

Received 16 January 2007; revision received 19 January 2007; accepted for publication 16 February 2007. Copyright © 2007 by TeK Associates. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission. Copies of this paper may be made for personal or internal use, on condition that the copier pay the \$10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 0731-5090/07 \$10.00 in correspondence with the CCC.

*9 Meriam Street, Suite 7-R; thomas_h_kerr@msn.com. Senior Member AIAA/IEEE.

(where details of accomplishing the MSF here are provided in [2], Appendix B) and for the resulting associated linear system realization:

$$\begin{aligned} \frac{d}{dt}x(t) &= F_1x(t) + G_1w'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 \end{bmatrix}x(t) \\ &+ \begin{bmatrix} -1 & 0 \\ (6-\sqrt{7})/2 & -1/2 \\ -1 & 0 \\ (6-\sqrt{7})/2 & 3/2 \end{bmatrix}w'(t) \end{aligned} \quad (3)$$

and

$$y(t) = H_1x(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}x(t) \quad (4)$$

which is of the form

$$\frac{d}{dt}x(t) = Fx(t) + Gw'(t) \quad \text{and} \quad y(t) = Hx(t) \quad (5)$$

and the corresponding $Q \triangleq GG^T$. Equation A1 in [1] and Eq. (5) here differ slightly in that the zero mean Gaussian white noise process $w'(t)$ in Eq. (5) has the matrix identity as its covariance intensity matrix but, otherwise, corresponds to the same second-order statistics for y as would be associated with Eq. A1, where $y(t) \equiv x(t)$ for Eq. A1 (corresponding to H being the identity matrix). This asserted solution as the linear system realization can easily be confirmed to yield the matrix power spectrum of Eq. (1) merely by using the right-hand expression of Eq. (2) and multiplying out the results with the asserted parameter values of Eqs. (3) and (4) to again yield the left-hand expression in Eq. (2), which, when further multiplied out as $W^T(-s)W(s)$ yields the power spectral matrix of Eq. (1) as a crosscheck.

We now attempt to apply the steps for achieving a linear system realization provided in [1] to the correlation function specified in Eq. (1), where it is more convenient to use the time domain representation in Eq. (1) instead. In attempting to take the derivative of the correlation function of Eq. (1) to be evaluated at $\tau = 0$ (corresponding to $\lim_{t \rightarrow t_0}$), as called for in Eq. A5, yields

[†]Precise regularity conditions guaranteeing that the steady-state constant symmetric positive definite matrix solution P_x can be obtained from [1] (Eq. A7) with $dP_x(t)/dt \equiv 0$ is that F have only eigenvalues with real parts strictly negative and that (F, L) be a controllable pair, where $Q = LL^T$ (and where L is a factor resulting from a Choleski decomposition of Q) [4].

[‡]In using this form, at this point in the derivation in [1], the assumption of stationarity had not yet been invoked as a slight misstep of prematurely using an assumption that is invoked later.

[§]The power spectrum matrix depicted here on the right-hand side of Eq. (1) consists of elements represented in the frequency domain by the bilateral Laplace transform, which relates to the Fourier transform via the substitution $s = j\omega$.

$$\lim_{\tau \rightarrow 0} \frac{d}{d\tau} R_{yy}(\tau) = \frac{d}{d\tau} \left[\begin{array}{ccc} \frac{1}{6}e^{-2|\tau|} + \frac{1}{6}e^{-|\tau|} & \cdots & \frac{1}{4}e^{-2|\tau|} \\ \cdots & \cdots & \cdots \\ \frac{1}{4}e^{-2|\tau|} & \cdots & \frac{1}{2}e^{-|\tau|} \end{array} \right] \Big|_{\tau=0} \quad (6)$$

but this presents a problem because the terms constituting the elements of the matrix, as a composite function, are not differentiable at $\tau = 0$ because the absolute value of τ , being $|\tau|$, itself is not differentiable at the origin and we are stymied by being unable to proceed any further using this approach. However, to illustrate what further problems are to be encountered, suppose that an even more benign matrix correlation function were being used such as that corresponding to the familiar *ergodic* random process ([6], Chap. 9) with both channels being independent, as a possible further simplification, then we have

$$x_1(t) = A \sin(\omega t + \theta), \quad x_2(t) = B \sin(\omega t + \theta) \quad (7)$$

with A , B , and ω being deterministic nonzero constants, and θ being a zero mean random variable uniformly distributed over $[-\pi, \pi]$. Its matrix correlation function would be

$$R_{yy}(\tau) = \begin{bmatrix} A \cos(\omega\tau) & 0 \\ 0 & B \cos(\omega\tau) \end{bmatrix} \quad (8)$$

and, upon attempting to differentiate it with respect to τ at $\tau = 0$, this step can now be accomplished as

$$\lim_{\tau \rightarrow 0} \frac{d}{d\tau} R_{yy}(\tau) = \left[\begin{array}{cc} -A\omega \sin(\omega\tau) & 0 \\ 0 & -B\omega \sin(\omega\tau) \end{array} \right] \Big|_{\tau=0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (9)$$

and we are one step further but stuck again and unable to take the approach of [1] to fruition. These two numerical examples are extremely well behaved and exhibit all the properties of a valid correlation function matrix ([3], Sec. IIB); however, they reveal the weaknesses of this asserted new approach although the initial covariances can be found for the preceding two examples, respectively, as

$$\begin{aligned} \lim_{\tau \rightarrow 0} R_{yy}(\tau) &= \lim_{\tau \rightarrow 0} \begin{bmatrix} \frac{1}{6}e^{-2|\tau|} + \frac{1}{6}e^{-|\tau|} & \cdots & \frac{1}{4}e^{-2|\tau|} \\ \cdots & \cdots & \cdots \\ \frac{1}{4}e^{-2|\tau|} & \cdots & \frac{1}{2}e^{-|\tau|} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \cdots & \frac{1}{4} \\ \cdots & \cdots & \cdots \\ \frac{1}{4} & \cdots & \frac{1}{2} \end{bmatrix} \end{aligned} \quad (10)$$

and as

$$\lim_{\tau \rightarrow 0} R_{yy}(\tau) = \lim_{\tau \rightarrow 0} \begin{bmatrix} A \cos(\omega\tau) & 0 \\ 0 & B \cos(\omega\tau) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad (11)$$

Because the means of both examples are zero vectors, here we have $R_{yy}(0) \triangleq E[x(t_o)x^T(t_o)] = P_x(t_o)$ and in both examples the resulting $P_x(t_o)$ is nonsingular and invertible.

Looking further at Eq. A5 [1] for what is needed to continue with the preceding two examples to specify a corresponding F using the newly posed methodology, even with the matrix $P_x(t_o)$ specified and nonsingular in both cases, one cannot proceed in specifying F from Eq. A5 and, consequently, Eq. A8 is useless as well. However, if one were attempting to solve Eq. A7 for the necessary constant $P_x(t_o)$ using the standard prior approach, one first needs to know F and Q . Although it is true that for a Kalman filtering situation, where any symmetric positive definite matrix suffices for the initial condition covariance to start integrating out the Riccati equation, which exponentially asymptotically converges to the correct solution under fairly mild regularity conditions [7] even if the exact initial covariance matrix is unknown, an arbitrary symmetric positive definite $P_x(t_o)$ hypothesized starting value does not necessarily correspond to the steady-state solution of Eq. A7 (and the rate of convergence in just integrating it out with time is much slower than that of the somewhat similar looking Riccati equation) to obtain the necessary final result for $P_x(t_o)$ that is needed to represent the corresponding stationary random process. Hence, one cannot use Eq. A5 to determine F and eventually Q from Eq. A8 for even these two benign examples, which points to a rather severe limitation in applicability of the new approach offered in [1] (Appendix).

Acknowledgment

Research funded by TeK Associates' Internal Research and Development Contract No. 07-101; www.TeKAssociates.biz.

References

- [1] Jekeli, C., "Precision Free-Inertial Navigation with Gravity Compensation by an Onboard Gradiometer," *Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 3, May–June 2006, pp. 704–713.
- [2] Kerr, T. H., "Multichannel Shaping Filter Formulations for Vector Random Process Modeling Using Matrix Spectral Factorization," Massachusetts Inst. of Technology Lincoln Lab., Rept. No. PA-500, Lexington, MA, 27 March 1989 (Ballistic Missile Organization limited distribution).
- [3] Kerr, T. H., "Emulating Random Process Target Statistics (using MSF)," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 30, No. 2, April 1994, pp. 556–577.
- [4] Kerr, T. H., "Testing Matrices for Positive Definiteness/Semi-definiteness and Application Examples that Spawn the Need," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 5, Sept.–Oct. 1987, pp. 503–506.
- [5] Sage, A. P., *Optimum Systems Control*, Prentice–Hall, Englewood Cliffs, NJ, 1968.
- [6] Schwarz, R. J., and Friedland, B., *Linear Systems*, McGraw–Hill, New York, 1965.
- [7] Aasneas, H. B., and Kailath, T., "Initial-Condition Robustness of Linear Least Squares Filtering Algorithms," *IEEE Transactions on Automatic Control*, Vol. 19, No. 4, Aug. 1974, pp. 393–397.